Analytic study of the urn model for separation of sand

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We present an analytic study of the urn model for separation of sand recently introduced by Lipowski and Droz (Phys. Rev. E **65**, 031307 (2002)). We solve analytically the master equation and the first-passage problem. The analytic results confirm the numerical results obtained by Lipowski and Droz. We find that the stationary probability distribution and the shortest one among the characteristic times are governed by the same *free energy*. We also analytically derive the form of the critical probability distribution on the critical line, which supports their results obtained by numerically calculating Binder cumulants (cond-mat/0201472).

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I. INTRODUCTION

A granular system exhibits extremely rich phenomena, which has recently attracted extensive studies. One of such interesting phenomena is the spatial separation of shaken sand [1]. Sand in a box separated into two equal parts by a wall that allows the transfer of sand through its narrow slit prefers to aggregate more in one side under certain conditions.

Eggers explained the emergence of symmetry breaking using a hydrodynamic approach [2]. The key idea is to introduce the effective temperature taking into account the inelastic collisions for granular material.

Lipowski and Droz proposed a dynamic model to explain the essence of the phenomena [3]. The model is a certain generalization of the Ehrenfest model [4]. Interestingly this model shows a spontaneous symmetry breaking in contrast to other generalizations of the Ehrenfest model. They derived the master equation and found in a numerical way the phase diagram that displays a rich structure like continuous and discontinuous transitions as well as a tricritical point. They also numerically solved the first-passage problem to find exponential or algebraic divergences.

Thanks to its simplicity, the model allows analytic approaches. In this paper, we present the results of this analytic study to the master equation and the first-passage problem addressed by Lipowski and Droz. These not only confirm their numerical results but also give us some insights in the nature of the discontinuous transition in the stationary probability distribution. We also analytically derive the form of the critical probability distribution on the critical line.

The paper is organized as follows. In Sec. II we briefly review the model and its master equation. In Sec. III we present the analytic solution of the master equation in the thermodynamic limit and the analytic expression of the stationary probability distribution. The form of the stationary probability distribution on the critical line is also derived. In Sec. IV we analytically solve the first-passage problem. Detailed analysis on the behavior of the characteristic times is given in Sec. V. Section VI is devoted to conclusions and discussions.

II. MODEL AND ITS MASTER EQUATION

The model introduced by Lipowski and Droz [3] is defined as follows. N particles are distributed between two urns, and the number of particles in each urn is denoted as M and $N-M$, respectively. At each time of updates one of the N particles is randomly chosen. Let n be a fraction of the total number of particles in the urn that the selected particles belongs to. With probability $\exp(-\frac{1}{T(n)})$ the selected particle moves to the other urn. $T(n)$ represents the effective temperature of an urn with particles nN that measures the thermal fluctuations of the urn. Lipowski and Droz chose the temperature as $T(n) = T_0 + \Delta(1-n)$.

It is easy to derive the master equation for the probability distribution $p(M,t)$ that there are M particles in a given urn at time t [3]

$$\begin{aligned} p(M,t+1) &= F\Big(\frac{N-M+1}{N}\Big)p(M-1,t) \\ &\quad + F\Big(\frac{M+1}{N}\Big)p(M+1,t) \\ &\quad + \Big[1 - F\Big(\frac{M}{N}\Big) - F\Big(\frac{N-M}{N}\Big)\Big]p(M,t), \end{aligned} \tag{1}$$

where $F(n) = n\exp(-\frac{1}{T(n)})$ measures the flux of particles leaving the given urn. Here we introduced for convenience the notations $p(-1,t) = p(N+1,t) = 0$.

The difference in the occupancy of the urns can be represented by the particle excess $\epsilon = \frac{M}{N} - \frac{1}{2}$. The time evolution of the averaged particle excess $e(t) = \langle\epsilon\rangle_t = \sum_M(\frac{M}{N} - \frac{1}{2})p(M,t)$ is governed by

$$e(t+1) = e(t) + \frac{1}{N}\big\langle\mathcal{F}(\epsilon)\big\rangle_t\,, \tag{2}$$

where $\mathcal{F}(\epsilon) = F(\frac{1}{2}-\epsilon) - F(\frac{1}{2}+\epsilon)$ measures the net flux of particles in the given urn. One conventionally takes the

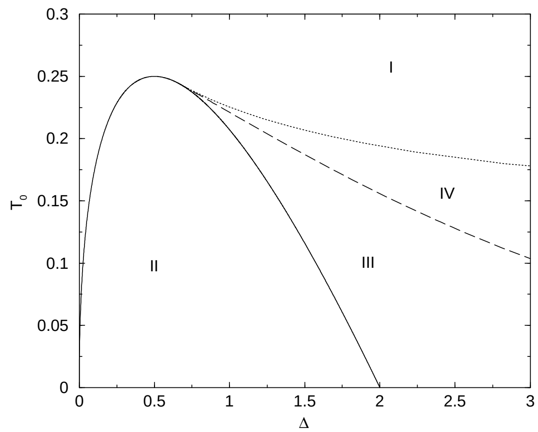

FIG. 1: Phase diagram of the urn model [3]. The symmetric solution vanishes continuously on the solid line while the asymmetric one disappears discontinuously on the dotted line. The transition of the behavior of the stationary probability distribution is denoted by the dashed line.

unit of time in such a way that there is a single update per a particle on average. Therefore we scale the time by N. Expanding Eq. (2) with respect to $\frac{1}{N}$, and using the mean–field approximation in evaluating the average, we get

$$\frac{d}{dt}e(t) = \mathcal{F}\big(e(t)\big)\,. \tag{3}$$

Note that the stationary solution of Eq. (3) is determined by zero points of $\mathcal{F}(\cdot)$. The stable stationary solutions are given by zero points of $\mathcal{F}(\cdot)$ with a negative slope, which we call as the stable fixed points while the unstable ones are given by those with a positive slope, which we call as the unstable fixed points.

Detailed analysis on the existence of the stable stationary solutions of Eq. (3) was done by Lipowski and Droz [3]. We here display their phase diagram in Fig. 1 to make our paper as self-contained as possible. The stable symmetric solution ($\epsilon = 0$) exists in region I, III, and IV while the stable asymmetric solution ($\epsilon > 0$) exists in region II, III and IV.

III. THE SOLUTION OF THE MASTER EQUATION

We are mainly interested in investigating the properties of the infinite system. Consider the thermodynamic limit $N \to \infty$ with $\frac{M}{N} = \frac{1}{2} + \epsilon$ being fixed. Representing the probability distribution by ϵ instead of M, and expanding Eq. (1) with respect to $\frac{1}{N}$, and keeping the terms up to the first order, we arrive at the expression

$$\begin{aligned} p(\epsilon, t+1) \;=\; & p(\epsilon, t) + \frac{1}{N}\Big[\big(F'(\frac{1}{2}+\epsilon) + F'(\frac{1}{2}-\epsilon)\big)p(\epsilon, t) \\ & + \big(F(\frac{1}{2}+\epsilon) - F(\frac{1}{2}-\epsilon)\big)\frac{\partial}{\partial\epsilon}p(\epsilon, t)\Big]\,. \end{aligned} \tag{4}$$

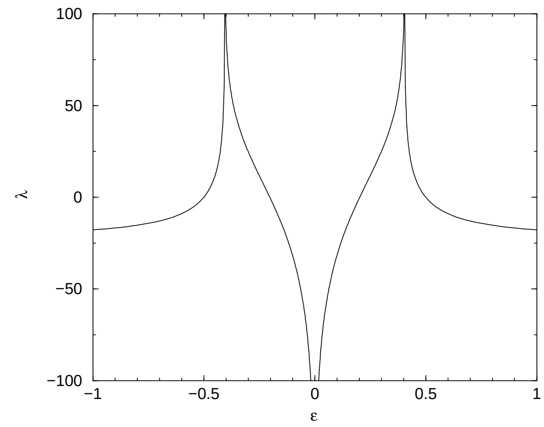

FIG. 2: $\lambda(\epsilon)$ for $\Delta = 0.3, T_0 = 0.2$. The mapping for $\epsilon > \frac{1}{2}$ is analytically continuated.

Scaling again the time by N, expanding Eq. (4) with respect to $\frac{1}{N}$, and noting that the second term in the right-handed side can be combined into a total derivative with respect to ϵ, we finally obtain the partial differential equation

$$\frac{\partial}{\partial t}p(\epsilon, t) + \frac{\partial}{\partial\epsilon}\big[\mathcal{F}(\epsilon)p(\epsilon, t)\big] = 0\,. \tag{5}$$

Note that $\mathcal{F}(\cdot)$ is zero at a finite number of points. The solution of Eq. (5) can be found in the intervals that do not include those points. At each interval, it would be convenient to introduce a new variable

$$\lambda(\epsilon) = \int_{\epsilon_0}^{\epsilon} \frac{dx}{\mathcal{F}(x)}\,, \tag{6}$$

where ϵ_0 is a certain point in the interval. Figure 2 shows a typical behavior of the mapping. We also displayed the map for $|\epsilon| > \frac{1}{2}$ by analytic continuation. This is necessary since the solution of Eq. (5) is of wave nature. (See Eq. (8) below.) λ increases as ϵ approaches to the stable fixed points while it decreases as ϵ approaches to the unstable fixed points. With the help of this parameterization and setting $R(\lambda, t) = \mathcal{F}(\epsilon)p(\epsilon, t)$, Eq. (5) now takes the form

$$\frac{\partial}{\partial t}R(\lambda, t) + \frac{\partial}{\partial\lambda}R(\lambda, t) = 0\,. \tag{7}$$

Note that Eq. (7) is in fact a half part of the wave equations so that its solution is written as $R(\lambda, t) = f(\lambda - t)$ with $f(\cdot)$ being an arbitrary differentiable function. It represents a wave that moves to the direction of increasing λ as time evolves, which means that the system moves to stable fixed points. The solution of the original partial differential equation (5) now reads

$$p(\epsilon, t) = \frac{f\big(\lambda(\epsilon) - t\big)}{\mathcal{F}(\epsilon)}\,. \tag{8}$$

From the initial probability distribution $p_0(\epsilon) = p(\epsilon, 0)$ we can determine the function $f(\cdot)$. So we get

$$p(\epsilon, t) = \frac{\mathcal{F}(\epsilon_t)}{\mathcal{F}(\epsilon)} p_0(\epsilon_t)\,, \tag{9}$$

where ϵ_t is given by the relation

$$\lambda(\epsilon_t) = \lambda(\epsilon) - t\,. \tag{10}$$

Here it should be understood that ϵ_t is to be chosen in the same interval where ϵ belongs to. Furthermore $p_0(\epsilon) = 0$ for $|\epsilon| > \frac{1}{2}$ is assumed since ϵ_t in Eq. (9) can be larger (or smaller) than $\frac{1}{2}$ (-$\frac{1}{2}$). This happens because of the nature of the wave solution. Using the mapping from ϵ to ϵ_t, it is straightforward to show that the total probability is conserved:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} p(\epsilon, t) d\epsilon = \int_{-\frac{1}{2}}^{\frac{1}{2}} p_0(\epsilon_t) d\epsilon_t = 1\,. \tag{11}$$

The shape of the probability distribution is distorted by a ratio $\mathcal{F}(\epsilon_t)/\mathcal{F}(\epsilon)$ so that it accumulates at the nearest stable fixed points. In Eq. (9), the ratio approaches zero as time evolves unless ϵ is on a stable fixed point. As a consequence, in the long time limit $t \to \infty$ the probability distribution becomes a sum of delta peaks at the stable fixed points ϵ_i

$$p(\epsilon, \infty) = \sum_i p_i \delta(\epsilon - \epsilon_i)\,, \tag{12}$$

where p_i are the sum of the initial probabilities in two intervals adjacent to its point ϵ_i. We would like to point out that the system is not ergodic and its dynamical phase space is decomposed into disconnected sectors. Each sector is associated with a stable fixed point and is separated by the unstable fixed points.

The fixed point condition $\mathcal{F}(\epsilon) = 0$ is equivalent to Eq. (4) in Ref. [3], where Lipowski and Droz analyzed in detail the condition and their results are summarized in the phase diagram (See Fig. 1.). We would like to mention that in regions III and IV in Fig. 1 both the symmetric and the asymmetric solutions exist together. In fact, either solution can be realized by choosing an appropriate initial configuration. Lipowski and Droz distinguished regions III and IV according to the different behaviors of the stationary probability distribution in their numerical process of taking the limit $N \to \infty$. In region III there appear two delta peaks for the asymmetric solutions while in region IV there appears only the central delta peak for the symmetric solution. It is contradictory to our result Eq. (12) where any delta peaks for the stable fixed points can appear depending on the initial configurations.

To resolve this contradiction and understand the nature of the transition between regions III and IV, we take another limit in the master equation (1), namely take the long time limit $t \to \infty$ before we take the limit $N \to \infty$. This limit may not properly reflect the properties of the infinite system. Since the infinite system is not ergodic as we showed above, changing the order of taking limits $N \to \infty$ and $t \to \infty$ may not yield the same result. Anyway it seems that in their simulations about the stationary probability distribution, Lipowski and Droz took the limit $t \to \infty$ for a finite system size N, and then extrapolating the results to $N \to \infty$.

Let's first take the long time limit of $t \to \infty$ in Eq. (1). In this limit we may drop off the time dependence in the probability distribution, which now takes the form

$$\begin{aligned} p(N) &= F\big(\frac{1}{N}\big)p(N-1) + \big(1 - F(1)\big)p(N) \\ p(M) &= F\big(\frac{N-M+1}{N}\big)p(M-1) \\ &\quad + F\big(\frac{M+1}{N}\big)p(M+1) \\ &\quad + \Big[1 - F\big(\frac{M}{N}\big) - F\big(\frac{N-M}{N}\big)\Big]p(M) \\ &\quad \text{for } M = N-1, \ldots, 2, 1 \\ p(0) &= F\big(\frac{1}{N}\big)p(1) + \big(1 - F(1)\big)p(0)\,. \end{aligned} \tag{13}$$

The first equation in Eq. (13) allows us to rewrite $p(N)$ in terms of $p(N-1)$, which in turn allows to rewrite $p(N-1)$ in terms of $p(N-2)$, and so on. Therefore we find

$$\begin{aligned} p(M) &= \frac{F\big(\frac{N-M+1}{N}\big)}{F\big(\frac{M}{N}\big)} p(M-1) \\ &= p(0) \prod_{i=1}^{M} \frac{F\big(\frac{N-i+1}{N}\big)}{F\big(\frac{i}{N}\big)} \end{aligned} \tag{14}$$

for $M = N, \ldots, 2, 1$. $p(0)$ appears as an overall factor to normalize the probabilities so that we get

$$p(0) = \Big[1 + \sum_{M=1}^{N} \prod_{i=1}^{M} \frac{F\big(\frac{N-i+1}{N}\big)}{F\big(\frac{i}{N}\big)}\Big]^{-1}. \tag{15}$$

Now let's take the limit $N \to \infty$. With $\frac{M}{N} = \frac{1}{2} + \epsilon$, and $\frac{i}{N} = \frac{1}{2} + x$, and scaling the probability distribution by N, the stationary probability distribution for large N now becomes

$$p_s(\epsilon) \approx \frac{e^{NG(\epsilon)}}{\int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, e^{NG(x)}}\,, \tag{16}$$

where

$$G(\epsilon) = \int_{-\frac{1}{2}}^{\epsilon} dx \left[\log F\big(\frac{1}{2} - x\big) - \log F\big(\frac{1}{2} + x\big)\right]. \tag{17}$$

In the limit $N \to \infty$, the main contribution to the stationary probability distribution comes only from the maximum of $G(\cdot)$, and it becomes delta peaks. The maximum of $G(\epsilon)$ occurs when

$$G'(\epsilon) = \log F\big(\frac{1}{2} - \epsilon\big) - \log F\big(\frac{1}{2} + \epsilon\big) = 0\,. \tag{18}$$

Since $G'(\epsilon)$ is the difference of logarithms of $F(\frac{1}{2}-\epsilon)$ and $F(\frac{1}{2}+\epsilon)$, both $G(\cdot)$ and $\mathcal{F}(\cdot)$ share the similar qualitative properties. For example, the maximum of both functions occurs at the same stable fixed points. Note that in region II only the two asymmetric solutions with positive and negative particle excesses are stable, and have the same maximum while in region I only the symmetric solution is stable. Therefore the stationary probability distribution has the double peaks in region II and only the central peak in region I. In region III and IV both the symmetric and the asymmetric solutions are stable so that the maximum of $G(\epsilon)$ should be determined by comparing its values at the stable fixed points. The crossover of the maximum point occurs when both values coincide. This implies that the transition between the double peaks and the central single peak in the probability distribution is determined by the condition $\Delta G = G(\epsilon_a) - G(0) = 0$, where ϵ_a is the nonzero stable fixed point. This condition yields a line that separate two regions III and IV.

It is very interesting to see that $-G(\cdot)$ resembles the free energy of the equilibrium systems, and the transition between two *phases* is determined by the condition that the free energies of both phases are equal. Furthermore a certain characteristic time behaves differently in two phases, as Lipowski and Droz numerically found [3]. We will show an analytic relation between them in next section.

IV. CHARACTERISTIC TIMES

Lipowski and Droz defined an averaged first-passage time $\tau(M)$ needed for a configuration with M particles in an urn (and $N-M$ particles in the other urn) to reach the symmetric configuration ($M=\frac{N}{2}$) [3]. They obtained the relations among the averaged characteristic times from the dynamical rules as

$$\begin{aligned}\tau(M) &= F\big(\frac{M}{N}\big)\big[\tau(M-1)+1\big] \\ &\quad + F\big(\frac{N-M}{N}\big)\big[\tau(M+1)+1\big] \\ &\quad + \Big[1-F\big(\frac{M}{N}\big)-F\big(\frac{N-M}{N}\big)\Big]\big[\tau(M)+1\big] \end{aligned} \quad (19)$$

for $M = N, N-1, \ldots, \frac{N}{2}+1$. Here it is understood that the term associated with $\tau(N+1)$ does not appear. (In fact, its coefficient $F(0)$ vanishes.) By definition of the characteristic times, $\tau(\frac{N}{2})=0$ and $\tau(N-M)=\tau(M)$.

Defining the difference of successive characteristic times as $\Delta\tau(M) = \tau(M)-\tau(M-1)$, Eq. (19) can be rewritten as

$$\Delta\tau(M) = \frac{1}{F\big(\frac{M}{N}\big)}\Big[1+F\big(\frac{N-M}{N}\big)\Delta\tau(M+1)\Big]. \quad (20)$$

By applying this relation repeatedly until $\Delta\tau(N) = \frac{1}{F(1)}$ is reached, we get the expression

$$\Delta\tau(M) = \frac{1}{F\big(\frac{M}{N}\big)}\Big[1+\sum_{i=1}^{N-M}\prod_{j=1}^{i}\frac{F\big(\frac{N-M-j+1}{N}\big)}{F\big(\frac{M+j}{N}\big)}\Big]. \quad (21)$$

Since $\tau(\frac{N}{2})=0$ by definition, we immediately get $\tau(\frac{N}{2}+1) = \Delta\tau(\frac{N}{2}+1)$, which is given by Eq. (21) with $M=\frac{N}{2}+1$. By successively adding $\Delta\tau(M)$, we get the general expression for $\tau(M)$ for $M=N, N-1, \ldots, \frac{N}{2}+1$:

$$\tau(M) = \sum_{k=\frac{N}{2}+1}^{M}\frac{1}{F\big(\frac{k}{N}\big)}\Big[1+\sum_{i=1}^{N-k}\prod_{j=1}^{i}\frac{F\big(\frac{N-M-j+1}{N}\big)}{F\big(\frac{M+j}{N}\big)}\Big]. \quad (22)$$

We are mainly interested in the behavior of $\tau(M)$ as N increases. Again we scale the characteristic times by N. With $\frac{M}{N}=\frac{1}{2}+\epsilon$ being fixed, and introducing $\frac{k}{N}=\frac{1}{2}+x$, $\frac{i}{N}=y-x$, $\frac{j}{N}=z-x$, the summations for large N can be replaced by integrations so that the characteristic times for $\epsilon>0$ takes the form

$$\tau(\epsilon) \approx N\int_0^{\epsilon}dx\int_x^{\frac{1}{2}}dy\, e^{NH(x,y)} \quad (23)$$

with $H(x,y)=G(y)-G(x)$. The longest characteristic time $\tau(N)$ corresponds to $\tau(\epsilon=\frac{1}{2})$. The shortest one $\tau(\frac{N}{2}+1)$ corresponding to $\tau(\epsilon=0)$ is, in general, smaller than $\tau(\epsilon)$ with positive ϵ by an order of magnitude. It is necessary to deal with it separately. We get

$$\tau(\frac{N}{2}+1) \approx \int_0^{\frac{1}{2}}dy\, e^{NH(0,y)}. \quad (24)$$

Since $H(0,y)=G(y)-G(0)$, both the shortest characteristic time $\tau(\frac{N}{2}+1)$ and the stationary probability distribution $p_s(\epsilon)$ have essentially the same functional dependence on $G(\cdot)$. Therefore it is not surprising that behaviors of both quantities for large N are closely related. However it is not clear why they are.

V. ANALYSIS OF THE CHARACTERISTIC TIMES

We first consider the behavior of $\tau(\frac{N}{2}+1)$, the shortest one among the characteristic times. For large N, the main contribution to $\tau(\frac{N}{2}+1)$ comes from the maximum point y_m of $H(0,y)=G(y)-G(0)$, or $G(y)$, which was dealt in Sec. III.

In region I and IV the maximum occurs at $y_m=0$ corresponding to the symmetric configuration, while in region II and III it occurs at $y_m>0$ corresponding to the asymmetric configurations. Therefore the maximum is zero in region I and IV while it is positive in region II and III.

When $y_m = 0$, we may expand $H(0, y)$ around $y = 0$ to get

$$H(0,y) \approx -2\Big[1 - \frac{\Delta/2}{(T_0+\Delta/2)^2}\Big]y^2 - \frac{4}{3}\Big[1 - \frac{3(\Delta/2)^2}{(T_0+\Delta/2)^4}\Big]y^4 + \dots . \quad (25)$$

Therefore it yields $\tau(\frac{N}{2}+1) \sim N^{-\frac{1}{2}}$ as long as the coefficient of the first term in Eq. (25) is negative. This is the case in region I and IV. The coefficient vanishes when $T_0 = \sqrt{\frac{\Delta}{2}} - \frac{\Delta}{2}$, which corresponds to the critical line found by Lipowski and Droz [3]. On this line, we get

$$H(0,y) \approx -\frac{4}{3}(1 - \frac{3}{2}\Delta)y^4 - \frac{32}{15}(1 - \frac{5}{4}\Delta^2)y^6 + \dots , \quad (26)$$

which yields $\tau(\frac{N}{2}+1) \sim N^{-\frac{1}{4}}$ for $\Delta < \frac{2}{3}$, and $\tau(\frac{N}{2}+1) \sim N^{-\frac{1}{6}}$ at $\Delta = \frac{2}{3}$ corresponding to the tricritical point.

When $y_m > 0$, the maximum is positive so that $\tau(\frac{N}{2}+1) \sim N^{-\frac{1}{2}}e^{\alpha N}$ (with α being a positive constant), that is, it diverges exponentially.

Now let's investigate the behavior of the longest characteristic time $\tau(N)$ or $\tau(\epsilon = 1)$. Again the main contribution comes from the maximum point of $H(x, y)$ in region restricted by three lines $y = x, x = 0, y = \frac{1}{2}$. Note that $y \geq x$ in the region. Interestingly the maximum point (x_m, y_m) is closely related with the fixed points of $\mathcal{F}(\epsilon)$.

In region I, $\epsilon = 0$ is only the fixed point so that $x_m = y_m = 0$, and the maximum is zero. Expanding $H(x, y)$ about this point, we get

$$H(x,y) \approx -2\Big[1 - \frac{\Delta/2}{(T_0+\Delta/2)^2}\Big](y^2 - x^2) + \dots . \quad (27)$$

The coefficient of the leading term in Eq. (27) is negative only if $T_0 > \sqrt{\frac{\Delta}{2}} - \frac{\Delta}{2}$, that is, above the critical line. Changing the variables x, y to the polar coordinates r, θ and scaling the radial coordinate r by $\sqrt{N}$, we arrive at

$$\tau(N) \approx \int_0^{c\sqrt{N}} dr\, r \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta\, \exp\Big[-2\Big(1 - \frac{\Delta/2}{(T_0+\Delta/2)^2}\Big) \times r^2\cos(2\theta)\Big] . \quad (28)$$

Here c is a constant that gives the upper bound of the integration over r. The contribution from the neighborhood of $\theta = \frac{\pi}{4}$ yields a logarithmic divergence. Fig. 3 shows a typical behavior of characteristic time $\tau(N)$ as a function of N for several values of Δ with $T_0 = 0.2$. The first two uppermost lines stand for $\tau(N)$ in region II, and the others represent that in region I. We conclude that in region I $\tau(N)$ diverges logarithmically as N increases.

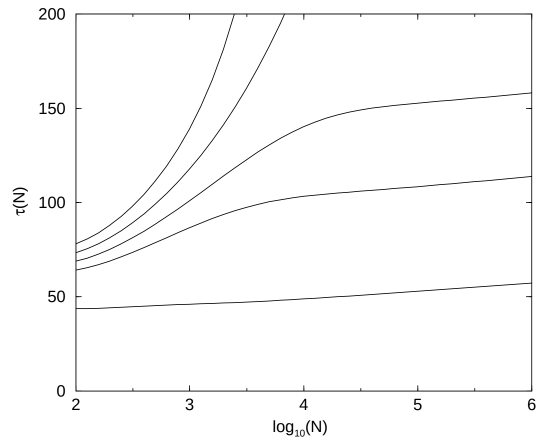

FIG. 3: characteristic time $\tau(N)$ as a function of N for $T_0 = 0.2$ and $\Delta = 1.72, 1.744067, 1.77, 1.8, 2.0$ (from the top).

As we see in Eq. (27), the leading term vanishes on the critical line. On this line we need to expand more. So

$$H(x,y) \approx -\frac{4}{3}(1 - \frac{3}{2}\Delta)(y^4 - x^4) - \frac{32}{15}(1 - \frac{5}{4}\Delta^2)(y^6 - x^6) + \dots . \quad (29)$$

Again changing the variables x, y to the polar coordinates and scaling the radial coordinate appropriately (by $N^{\frac{1}{4}}$ or $N^{\frac{1}{6}}$), we conclude that $\tau(N)$ diverges algebraically as $N^{\frac{1}{2}}$ on the critical line and as $N^{\frac{2}{3}}$ at the tricritical point.

In regions II, III, and IV there appear many fixed points among which we can always find one with $y_m > x_m$ and $G(y_m) > G(x_m)$. There, the maximum $H(x_m, y_m)$ is positive, and $\tau(N)$ diverges exponentially as N increases. We would like to point out that the situation is different from that of $\tau(\frac{N}{2}+1)$. In fact, it corresponds to the case with x_m being fixed to zero.

Finally let's consider the behavior of $\tau(N)$ on the line separating two regions I and IV. As we approach this line from region IV, the nonzero fixed points merge to disappear at $\epsilon = \epsilon_1 > 0$. That is, we can find a positive ϵ_1 such that $\mathcal{F}(\epsilon_1) = \mathcal{F}'(\epsilon_1) = 0$. On this line the maximum point is given by $x_m = y_m = \epsilon_1$ so that the maximum $H(x_m, y_m)$ is zero. Expanding $H(x, y)$ around this point, we get

$$H(x,y) \approx -\frac{1}{6}G'''(\epsilon_1)\big((y-\epsilon_1)^3 - (x-\epsilon_1)^3\big) + \dots . \quad (30)$$

(Here we don't write down $G'''(\epsilon_1)$ explicitly since it is not important as far as it is positive.) Note that the leading order is the third instead of the fourth as in case of the critical line. The reason is that $G(\epsilon)$ is not symmetric about $\epsilon = \epsilon_1$ while it is symmetric about $\epsilon = 0$. Consequently $\tau(N)$ on this line diverges algebraically as $N^{\frac{1}{3}}$.

VI. CONCLUSIONS

We analytically investigate the urn model introduced by Lipowski and Droz [3]. We exactly solve the master equation of the model in the thermodynamic limit and find how the probability distribution evolves. In the long time limit, the probability distribution becomes delta peaks only at the stable fixed points. In fact the ergodicity of the dynamics is broken so that the dynamical phase space is decomposed into disconnected sectors separated by the unstable fixed points. The strength of a delta peak is equal to the sum of initial probabilities in the disconnected sector it belongs to.

We also solve exactly the stationary probability distribution where we take the long time limit before we take thermodynamic limit. Regardless of the initial probability distribution it shows double peaks or a single central peak depending on the parameters of the system. The final formula of the stationary probability distribution resembles that of the equilibrium systems, where the transition from the doubles peaks to the single peak is determined by the condition that *free energies* of two phases become equal.

Recently Lipowski and Droz [5] numerically calculated Binder cumulants of the urn model to find that the critical probability distribution has the form $p(x) \sim e^{-x^4}$ on the critical line, and $p(x) \sim e^{-x^6}$ on the tricritical point, where x is the rescaled order parameter proportional to the particle excess ϵ. As we showed, $G(\epsilon) \sim H(0, \epsilon)$ and its behavior on the critical line (including the tricritical point) is given by Eq. (26). Therefore the critical probability distribution has the form $p(\epsilon) \sim \exp[-\frac{4}{3}(1-\frac{3}{2}\Delta)\epsilon^4]$ on the critical line, and $p(\epsilon) \sim \exp[-\frac{128}{135}\epsilon^6]$ on the tricritical point. Our analytic result supports their numerical result.

The first-passage problem is analytically solved. Interestingly both the shortest characteristic time and the stationary probability distribution are governed by the same *free energy*. Therefore the behavior of the shortest characteristic time and the properties of the stationary probability distribution are closely related. The analytic results on the behavior of the characteristic times support the numerical results of Lipowski and Droz [3].

Finally it would be very interesting to understand why both the stationary probability distribution and the shortest characteristic time are governed by the same *free energy*. It would also be of interest to extend our analytic study to many-urn models and other types of urn models.

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